Final Exam: MAT 203

Instructions: Complete all problems below. You may not use calculators or other aides, including cell phones and books. Show all of your work. **Be sure to write your name and student ID on each page that you hand in.**

1.(15pts) Consider the function $f(x,y) = y^2 + 2x$ along a constraint set defined by $x^2+y^2 = 6x$. Find the maximum and minimum values of f subject to this constraint, and find all points where the maxima and minima are achieved.

$\nabla f = (2, 2y)$, $\nabla g = (2x-6, 2y)$

where $g(x,y) = x^2+y^2-6x$ ← circle level set

Solve $\nabla f = \lambda \nabla g$, $g(x,y) = 0 \Leftrightarrow$

$$\begin{cases} 2 = \lambda(2x-6) \\ 2y = 2\lambda y \\ 6x = x^2+y^2 \end{cases} \Rightarrow 0 = 2y(\lambda - 1)$$

$\underline{y=0}$ or $\underline{\lambda = 1}$

$x^2 - 6x = 0$

$\Rightarrow x = 0$ or $x = 6$

$\Rightarrow \lambda = -\frac{1}{3}$ $\Rightarrow \lambda = \frac{1}{3}$

$\Rightarrow 2 = 2x - 6$

$\Rightarrow x = 4$

$\Rightarrow 6 \cdot 4 = 16 + y^2$

$\Rightarrow y = \pm\sqrt{8}$.

CP: $(0,0)$, $(6,0)$, $(4,\sqrt{8})$, $(4,-\sqrt{8})$

$\boxed{\underbrace{f(0,0) = 0}_{\text{min}}}$, $f(6,0) = 12$, $\boxed{\underbrace{f(4,\pm\sqrt{8}) = 16}_{\text{max}}}$.

2.(15pts) A wire in the xy-plane has the shape of a curve C parameterized by

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}, \qquad 0 \le t \le \frac{\pi}{2}.$$

The density of the wire is given by the function $\rho(x, y) = x + y$. Calculate the total mass of the wire.

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \sqrt{\sin^2 t + \cos^2 t}\, dt = dt$$

$$\text{Mass} = \int_C \rho\, ds = \int_0^{\pi/2} (\cos t + \sin t)\, dt$$

$$= (\sin t - \cos t) \Big|_0^{\pi/2}$$

$$= 1 - 0 - 0 + 1$$

$$= \boxed{2}$$

3.(20pts) Consider a surface S in $\mathbb{R}^3$ defined by the equation $z = xy$.

a) Write the equation of the tangent plane to S at the point $(x_0, y_0, z_0) = (1, 2, 2)$.

b) Calculate the surface area of S over the region $x^2 + y^2 \leq 1$.

a) Set $G(x,y,z) = xy - z$ then

$\nabla G = (y, x, -1)$ is normal to S.

$\nabla G(1,2,2) = (2, 1, -1)$

Tangent plane is $\boxed{2(x-1) + (y-2) - (z-2) = 0.}$

b) Let $g(x,y) = xy$ then the element of area is

$\sqrt{1 + g_x^2 + g_y^2}\, dx\, dy = \sqrt{1 + y^2 + x^2}\, dx\, dy$

$\Rightarrow$ Surface area $= \displaystyle\iint_{x^2+y^2 \leq 1} \sqrt{1+x^2+y^2}\, dx\, dy$

Polar coords. $\longrightarrow$ $= \displaystyle\int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\, r\, dr\, d\theta$

$= 2\pi \left.\dfrac{(1+r^2)^{3/2}}{3}\right|_0^1$

$= \boxed{2\pi\left(\dfrac{2^{3/2}}{3} - \dfrac{1}{3}\right)}$

4.(15pts) a) Calculate the volume of the solid region bounded by

$$0 \leq y \leq 1, \qquad 0 \leq x \leq y, \qquad 0 \leq z \leq xy.$$

b) Calculate the volume of the solid region bounded by the surface $z = 6 - x^2 - y^2$ and the xy-plane.

a) $$V = \int_0^1 \int_0^y xy \, dx \, dy = \int_0^1 \frac{x^2}{2} y \Big|_{x=0}^{y} dy$$

$$= \int_0^1 \frac{y^3}{2} dy = \frac{y^4}{8} \Big|_0^1 = \boxed{\frac{1}{8}}$$

b) $$V = \iint_{x^2+y^2 \leq 6} (6 - x^2 - y^2) \, dx \, dy$$

Polar coords $\rightarrow$ $$= \int_0^{2\pi} \int_0^{\sqrt{6}} (6 - r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(3r^2 - \frac{r^4}{4}\right) \Big|_0^{\sqrt{6}} d\theta$$

$$= 2\pi \left(3 \cdot 6 - \frac{6 \cdot 6}{4}\right)$$

$$= \boxed{18\pi}$$

5.(15pts) a) Check that the vector field $\mathbf{F}(x,y) = -(e^x \sin y)\mathbf{i} - (e^x \cos y)\mathbf{j}$ is conservative in the xy-plane.

b) Find a potential function for $\mathbf{F}$.

c) Calculate the work done by $\mathbf{F}$ on a particle that is moving in the plane from the point $(0,\pi)$ to the point $(1,2\pi)$ along a straight line.

a) $M = -e^x \sin y$, $N = -e^x \cos y$

$M_y = -e^x \cos y = N_x$ It is conservative.

b) Solve $f_x = -e^x \sin y$, $f_y = -e^x \cos y$.

Integrating the first equation yields

$f = -e^x \sin y + h(y)$

$\Rightarrow -e^x \cos y = f_y = -e^x \cos y + h'(y)$

$\Rightarrow h'(y) = 0 \Rightarrow h(y) = c$ constant.

$\Rightarrow \boxed{f(x,y) = -e^x \sin y + c}$

c) Work $= \int_C F \cdot dr = f(1,2\pi) - f(0,\pi)$

$= -e \sin(2\pi) + e^0 \sin(\pi) = \boxed{0}$.

6.(20pts) Use Green's theorem to evaluate

$$\int_C [y\cos(xy) + \ln(x+3)]\,dx + [x\cos(xy) + e^y + x]\,dy,$$

where C is the circle of radius 2 centered at $(0,4)$ and oriented counterclockwise.

$$M = y\cos(xy) + \ln(x+3)$$

$$N = x\cos(xy) + e^y + x$$

$$M_y = \cos(xy) - xy\sin(xy)$$

$$N_x = \cos(xy) - xy\sin(xy) + 1$$

$$\int_C M\,dx + N\,dy = \iint_{x^2+(y-4)^2 \le 4} \overbrace{(N_x - M_y)}^{1}\,dx\,dy$$

$$= \text{Area of disk}$$

$$= \boxed{4\pi}$$

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